Effect of Thrust on the Longitudinal Oscillations of Missiles

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OSWALD[1] obtained the linearized solution for the longitudinal oscillations of a missile being launched at a nearly constant acceleration by a constant thrust. Laitone and Lin[2] considered the case where the thrust increased as the speed increased so as to maintain a constant acceleration with a nearly horizontal trajectory. This analysis provided a linearized solution that predicted the longitudinal oscillations of a missile undergoing constant acceleration at any speed, from subsonic to hypersonic flight, as long as the aerodynamic coefficients were nearly constant for the duration of the short period oscillations in the angle of attack. The present paper provides the exact linearized solutions for the longitudinal oscillations when the trajectory is nearly horizontal and the thrust is either zero, a specified constant value, or increasing with the square of the increasing flight speed. The notation and flight path axes system used are identical to those given in Laitone and Lin.[2]

From Laitone and Lin[2] we have the following linearized differential equation for the angle of attack (α) oscillations with a varying thrust and a nearly horizontal trajectory (so that ρ and μ remain nearly constant)

$$d^2\alpha/dt^2 + b(t)\,d\alpha/dt + c(t)\alpha = 0 \tag{1}$$

$$b(t) = (V/L)\mu[C_{L\alpha} - \sigma(C_{m\mathring{\alpha}} + C_{mq})] + (T/mV) \tag{2}$$

$$c(t) = (V/L)^2\mu\sigma[-C_{m\alpha} - C_{mq}(\mu C_{L\alpha} + TL/mV^2)] + [(V/L)\mu C_{L\alpha} - (T/mV)](\mathring{V}/V) + \mathring{T}/mV \tag{3}$$

where

$$\mu = \rho SL/2m; \qquad \sigma = mL^2/I \tag{4}$$

and

$$(T/mV) = (\mathring{V}/V) + (V/L)\mu C_D \tag{5}$$

The variation of the thrust with time is given by $\mathring{T} = (dT/dt)$, and the resulting velocity variation $\mathring{V} = (dV/dt)$ can represent either an acceleration or a deceleration, as determined by Eq. (5). The physical parameters μ and σ can be considered to be constant, even if the missile is burning fuel very rapidly, because of the short period of the angle of attack oscillations. Barton's[3] calculations for the effect of the decrease in the missile's mass upon its longitudinal oscillations show only a negligible increase in the damping and oscillation frequency, even when the aerodynamic damping is relatively small and the motion is considered over many cycles.

For the special case of an increasing thrust that is applied at an initial speed $V = V_o$ so as to maintain a constant acceleration ($\mathring{V} = A$), Laitone and Lin[2] gave the solution of Eq. (1) in terms of the Coulomb wave functions, and for the usual case where $\mu < 10^{-2}$, $-C_{m\alpha} > 10^{-1}$ and

$$(\mu\sigma)^{1/2}(-C_{m\mathring{\alpha}})(-C_{m\alpha})^{-1/2} < 1$$

the oscillatory solution could be written as

$$\alpha(t)/\alpha_o = (1 + At/V_o)^{-1}\exp\{[-b_o/4][(1 + At/V_o)^2 - 1]\}\cdot\{\sin\lambda_o[(1 + At/V_o)^2 + \text{const}]\} \tag{6}$$

where

$$b_o = \mu V_o^2(AL)^{-1}[C_{L\alpha} + C_D - \sigma(C_{m\mathring{\alpha}} + C_{mq})] \tag{7}$$

$$\lambda_o \approx (1/2)V_o^2(AL)^{-1}[\mu\sigma(-C_{m\alpha})]^{1/2} \tag{8}$$

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Effect of Other Thrust Variations

It is evident from Eq. (6) that acceleration increases the damping of the longitudinal oscillations. Now it will be shown how any positive thrust improves the longitudinal dynamic stability. First we will consider the coasting missile with zero thrust that is being decelerated by its drag at the moment when its trajectory is nearly horizontal. This is not a special case of the coasting missile that is either descending or ascending since the differential equation that can be solved explicitly so as to include an exponential change in atmospheric density with altitude has infinite magnitudes for its coefficients when the trajectory is horizontal, e.g., see Allen[4] or Laitone and Lin.[2] Also, the coefficients $b(t)$ and $c(t)$ in Eq. (1) are complicated functions because $\mathring{V}(t)$ is not now constant as in Eq. (6), but instead is given by Eq. (5) with $T = 0$ as

$$dV/dt = \mathring{V} = -(V^2/L)\mu C_D \tag{9}$$

However, if we introduce the variable ξ defined by

$$\xi = L^{-1}\int V(t)\,dt; \qquad d\xi/dt = V(t)/L \tag{10}$$

$$d\alpha/dt = (V/L)(d\alpha/d\xi)$$

$$d^2\alpha/dt^2 = (V/L)^2[d^2\alpha/d\xi^2 + V^{-1}(dV/d\xi)(d\alpha/d\xi)]$$

then Eq. (1) is transformed to the following differential equation whose aerodynamic coefficients are now independent of $V(t)$ (so if the thrust is zero its dynamic coefficients are also constant)

$$d^2\alpha/d\xi^2 + b_2\,d\alpha/d\xi + c_2\alpha = 0 \tag{11}$$

where

$$b_2 = \mu[C_{L\alpha} - C_D - \sigma(C_{m\mathring{\alpha}} + C_{mq})] + 2TL(mV^2)^{-1} \tag{12}$$

$$c_2 = \mu[-\sigma C_{m\alpha} - \mu C_{L\alpha}(C_D + \sigma C_{mq})] + \mathring{T}L^2(mV^3)^{-1} + TL(mV^2)^{-1}[\mu(C_{L\alpha} + C_D - \sigma C_{mq}) - TL(mV^2)^{-1}] \tag{13}$$

The varying thrust terms have been included in the above equations so that later we can consider other cases. However, for the simplest case of the coasting missile with $T = 0$ we see that b_2 and c_2 are both constant for the nearly horizontal trajectory so the oscillatory solution of Eq. (11) is

$$\alpha(\xi)/\alpha_o = [\exp(-b_2/2)\xi][\sin\lambda_2(\xi + \text{const})] \tag{14}$$

where

$$\lambda_2 = [c_2 - (b_2/2)^2]^{1/2} \approx c_2^{1/2} \approx [\mu\sigma(-C_{m\alpha})]^{1/2} \tag{15}$$

for the usual case having $\mu < 10^{-2}$ and $-C_{m\alpha} > 10^{-1}$. However, in order to be able to compare Eq. (14) with Eq. (6) we must convert from ξ to t by integrating Eqs. (9) and (10), respectively, so as to obtain

$$V(t) = V_o(1 + \mu C_D V_o t/L)^{-1} \tag{16}$$

$$\xi(t) = (\mu C_D)^{-1}\ln(1 + \mu C_D V_o t/L) \tag{17}$$

The exponential damping term in Eq. (14) accordingly becomes

$$|\alpha|/\alpha_o = (1 + \mu C_D V_o t/L)^{-(b_2/2)(\mu C_D)^{-1}} \tag{18}$$

Although of a different form, the damping expressed in Eq. (18) will be shown to be similar to the typical exponential damping found in Eq. (6) for constant acceleration, and the constant velocity case as given by (e.g., see Laitone and Lin[2])

$$|\alpha|/\alpha_o = \exp(-b_1/2)t \tag{19}$$

where

$$b_1 = (\mu V_o/L)[C_{L\alpha} + C_D - \sigma(C_{m\mathring{\alpha}} + C_{mq})] \tag{20}$$

In the later constant velocity case the constant thrust exactly balances the drag so the thrust term in Eqs. (2) and (3) has been replaced by C_D in accordance with Eq. (5) when $\mathring{V} = 0$. Consequently the coefficients in Eq. (1) are now constant so Eqs. (19) and (20) are easily obtained.

Another exact solution of Eq. (11) can be obtained for the particular case wherein $T/V^2 =$ constant so that b_2 is again constant and c_2 is nearly constant except for the term $(\mathring{T}L^2/mV^3)$ which is extremely small for any finite velocity occurring after the initial launch period. Then the constant coefficient solution of Eq. (11) may be written as

$$\alpha(\xi)/\alpha_o = [\exp(-b_3/2)\xi][\sin\lambda_3(\xi + \text{const})] \tag{21}$$

where the constants are given by

$$b_3 = \mu[C_{L\alpha} + C_D - \sigma(C_{m\dot{\alpha}} + C_{mq})] + 2K \quad (22)$$

$$K = [TL(mV^2)^{-1} - \mu C_D] = (\dot{V}/V)(L/V) = V^{-1}\, dV/d\xi \quad (23)$$

$$\lambda_3 = [c_3 - (b_3/2)^2]^{1/2} \approx [\mu\sigma(-C_{m\alpha})]^{1/2} \quad (24)$$

$$c_3 = \{\mu[-\sigma C_{m\alpha} - \mu C_{L\alpha}(\sigma C_{mq} + C_D)] + (K + \mu C_D)[\mu(-\sigma C_{mq} + C_{L\alpha}) - K]\} \approx \mu\sigma(-C_{m\alpha}) \quad (25)$$

The approximations indicated in Eqs. (24) and (25) are valid when $K \ll 1$ for the usual missile having $\mu < 10^{-2}$ and $-C_{m\alpha} > 10^{-1}$.

Again in order to be able to compare the damping with Eq. (6) for constant acceleration, and with Eq. (18) for deceleration with zero thrust, we must integrate Eqs. (5) and (10), respectively, so as to obtain

$$V(t)/V_o = (1 - KV_o t/L)^{-1} \quad (26)$$

$$\xi(t) = K^{-1} \ln(1 - KV_o t/L)^{-1} \quad (27)$$

Consequently the exponential damping expressed in Eq. (21) is now transformed to

$$|\alpha|/\alpha_o = \exp(-b_3/2)\xi = [1 - KV_o t/L]^{(b_3/2K)} \quad (28)$$

Again the damping, when expressed in terms of t appears different in form than the usual exponential damping, however it is easily shown that all of these damping expressions have a similar effect on a typical missile. The reason for this can be best understood by comparing the initial slope (i.e., the tangent line approximation) for each of the examples we have studied at the initial time ($t = 0$) when the additional thrust is applied after the missile has attained the velocity $V_o > 0$. Then we have the following cases for the initial slope of the envelope of the longitudinal oscillations in angle of attack for a nearly horizontal flight path

($\dot{V} < 0$) coasting deceleration from Eq. (18) for $T = 0$

$$|\alpha|/\alpha_o \to [1 - (b_2/2)(V_o t/L)] = [1 - (b_1/2 - \mu C_D V_o/L)t] \quad (29)$$

Constant velocity from Eq. (19) for $T = (1/2)\rho V_o^2 SC_D = \text{const}$

$$|\alpha|/\alpha_o \to [1 - (b_1/2)t] \quad (30)$$

Constant acceleration from Eq. (6) with $T = mA + (1/2)\rho V^2 SC_D$

$$|\alpha|/\alpha_o \to [1 - (1 + b_o/2)(At/V_o)] = [1 - (b_1/2 + A/V_o)t] \quad (31)$$

$\ddot{V} > 0$ from Eq. (28) with $TL(mV^2)^{-1} = (K + \mu C_D) = \text{const}$

$$|\alpha|/\alpha_o \to [1 - (b_3/2)(V_o t/L)] = [1 - (b_1/2 + KV_o/L)t] \quad (32)$$

where b_1 and K are the constants given by Eqs. (20) and (23), respectively.

The interesting conclusion is that the initial slope of the envelope of the angle-of-attack oscillations is given by Eq. (32) for all of the above cases. For example, if $T = 0$ then $K = -\mu C_D$ and we obtain Eq. (29). Then if $K = 0$ we have the constant velocity case as given by Eq. (30), and finally, if $K = AL/V_o^2 = [TL(mV_o^2)^{-1} - \mu C_D]$ we find that Eq. (32) reduces to Eq. (31) for constant acceleration. Consequently the initial slope of the envelope of the angle-of-attack oscillations could be estimated by Eq. (32) for any thrust variation if one uses the instantaneous values of b_1 and K at the given velocity V_o. However, after a few seconds the damping provided by Eq. (28), which corresponds to $T/V^2 = \text{constant}$, is much greater than that for any of the other cases considered. This occurs because Eq. (26) shows that the acceleration is continually increasing, and as $t \to (L/KV_o)$ we have $|\alpha| \to 0$ as $V \to \infty$ and $\dot{V} \to \infty$. However, when $K = -\mu C_D$, corresponding to $T = 0$, Eqs. (21–28) become identical to the coasting case given by Eqs. (14–18), with $b_2 = b_3$, etc., since $T = 0$. Now there is no time limit on Eqs. (26) and (28) since $K < 0$ and the velocity is monotonically decreasing.

Although the preceding theoretical analysis was for a nearly horizontal trajectory, still the calculations of Oswald[1] indicate that Eq. (32) should be applicable to any accelerating missile in a climbing trajectory if the speed is not hypersonic, and the climb angle is less than 20°. However, at hypersonic speeds any finite upward or downward inclination of the trajectory necessitates the consideration of the atmospheric density variation as given in the theoretical studies by Allen[4] and Laitone and Lin.[2]

Effect of Thrust on Oscillation Frequency

It is evident that in all cases any increase in the thrust, or any acceleration along the nearly horizontal trajectory, will directly increase the damping of the angle-of-attack oscillations, as indicated by Eq. (32). However, the corresponding increase in the frequency of the α oscillations is at first negligible, and then increases very slightly with time unless V_o is very small. If we express the frequency of the α oscillations in terms of ω, then we have the following relations:

Constant acceleration from Eq. (6) with $T = mA + (1/2)\rho V^2 SC_D$

$$\omega(t) = (V_o/L)[\mu\sigma(-C_{m\alpha})]^{1/2}[1 + (1/2)(At/V_o)] \quad (33)$$

Increasing acceleration from Eq. (21) with $K = \text{constant}$

$$\omega(t) = (V_o/L)[\mu\sigma(-C_{m\alpha})]^{1/2}[1 + (1/2)(KV_o t/L) + (1/3)(KV_o t/L)^2 + \cdots] \quad (34)$$

Since $(KV_o/L) = (\dot{V}_o/V_o)$ from Eq. (23), therefore in the limit as $t \to 0$, Eq. (34) reduces to Eq. (33) which has its frequency increase linearly with time. This increase is seen to be very small at high speeds because the time involved in the damping of the α oscillations is only a few seconds. It is important to note for Eq. (34) that as long as $t < (L/KV_o)$ it predicts a relatively small increase in the α oscillation frequency even though Eq. (26) shows that the velocity is increasing rapidly to an infinite limit. Since we are primarily interested in the behavior of the α oscillations for the first few seconds, we can use the linear variation of ω as given by the first term containing t in Eq. (34). This corresponds to many oscillations of a typical missile having $K \ll 1$.

It should also be noted that when $T = 0$ (so that $K = -\mu C_D$) we find that Eq. (34) reduces to the α oscillation frequency given by Eq. (14) for the coasting missile that is being decelerated by its own drag. The frequencies given by Eqs. (33) and (34) provide excellent approximations for the usual missile having $\mu < 10^{-2}$, $-C_{m\alpha} > 10^{-1}$ and $(V^2/L) \gg \dot{V}$ so that the approximations indicated in Eqs. (15) and (24) are valid.

However, when V_o is very small, as in the launching phase of the rocket or missile, then we must replace Eq. (6) by another form of the solution of Eq. (1). In this case we have, for the same type of missile as considered in Eq. (6), the following:

$$\alpha(t) \sim (2L/At^2)^{1/2} \sin(\lambda_4 At^2/2L) \exp[(-b_4/2)(At^2/2L)] \quad (35)$$

where

$$\lambda_4 \approx [\mu\sigma(-C_{m\alpha})]^{1/2} \quad (36)$$

$$b_4 = \mu[C_{L\alpha} + C_D - \sigma(C_{m\dot{\alpha}} + C_{mq})] \quad (37)$$

This solution for constant acceleration corresponds to $V_o \to 0$ and can be obtained from Eq. (1) by the procedure given in Laitone and Lin.[2] It is identical to the solution used by Oswald[1] if we neglect the terms containing μ^2 (since $\mu < 10^{-2}$), as in the derivation for λ in Eqs. (15) and (24). The behavior of the type of angle-of-attack oscillations predicted by Eq. (35) during the launch phase are explicitly shown by Oswald[1] (page 790, Fig. 20).

References

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